

Generalized Kochen-Specker Theorem*

Asher Peres

Department of Physics, Technion – Israel Institute of Technology, 32 000 Haifa, Israel

Abstract

A generalized Kochen-Specker theorem is proved. It is shown that there exist sets of n projection operators, representing n yes-no questions about a quantum system, such that none of the 2^n possible answers is compatible with sum rules imposed by quantum mechanics. Namely, if a subset of commuting projection operators sums up to a matrix having only even or only odd eigenvalues, the number of “yes” answers ought to be even or odd, respectively. This requirement may lead to contradictions. An example is provided, involving nine projection operators in a 4-dimensional space.

* Dedicated to Professor Max Jammer, on the occasion of his 80th birthday.

The Kochen-Specker theorem [1] is of fundamental importance for quantum theory. In its original form, it asserts that, in a Hilbert space with a finite number of dimensions, $d \geq 3$, it is possible to produce a set of n projection operators, representing yes-no questions about a quantum system, such that none of the 2^n possible answers is compatible with the sum rules of quantum mechanics. Namely, if a subset of mutually orthogonal projection operators sums up to the unit matrix, one and only one of the corresponding answers ought to be yes. This requirement cannot be fulfilled. The physical meaning of this theorem is that there is no way of introducing noncontextual “hidden” variables [2] which would ascribe definite outcomes to these n yes-no tests. This conclusion holds irrespective of the preparation (the quantum state) of the system being tested.

It is also possible to formulate a “state-specific” version of this theorem, valid for systems that have been prepared in a known pure state. In that case, the projection operators are chosen in a way adapted to the known state. A smaller number of questions is then sufficient to obtain incompatibility with the quantum mechanical sum rules. An even smaller number is needed if strict sum rules are replaced by weaker probabilistic arguments [3, 4].

The original proof by Kochen and Specker [1] involved projection operators over 117 vectors in a 3-dimensional real Hilbert space \mathbb{R}^3 . A simple proof with 33 vectors was later given by Peres [5], who also reported an unpublished construction by Conway and Kochen, using only 31 vectors [6]. A proof with 20 vectors in \mathbb{R}^4 was given by Kernaghan [7], and one with 30 projection operators in \mathbb{R}^8 by Kernaghan and Peres [8].

There also is a multiplicative variant of the Kochen-Specker theorem, first mentioned by Fine and Teller [9]. Some explicit examples were given by Mermin [10], after a particular state-specific case was discovered by Peres [11]. The simplest one of these examples involves nine operators in four dimensions, whose eigenvalues are 1, 1, -1 , and -1 . They can be written in terms of Pauli matrices for a pair of spin- $\frac{1}{2}$ particles, as follows:

$$\begin{array}{lll}
 \mathbf{1} \otimes \sigma_z & \sigma_z \otimes \mathbf{1} & \sigma_z \otimes \sigma_z \\
 \sigma_x \otimes \mathbf{1} & \mathbf{1} \otimes \sigma_x & \sigma_x \otimes \sigma_x \\
 \sigma_x \otimes \sigma_z & \sigma_z \otimes \sigma_x & \sigma_y \otimes \sigma_y
 \end{array} \tag{1}$$

Each one of these nine operators has eigenvalues ± 1 . In each row and in each column, the three operators commute, and each operator is the product of the two others, *except*

in the third column, where an extra minus sign is needed:

$$(\sigma_x \otimes \sigma_x)(\sigma_y \otimes \sigma_y) = -(\sigma_z \otimes \sigma_z). \quad (2)$$

Because of that minus sign, it is impossible to attribute to the nine elements of the above array numerical values, 1 or -1 , which would be the results of measurements of these operators (if such measurements were performed), and which would obey the same multiplicative rule as the operators themselves. We have reached a contradiction.

We thus see that what we call “the result of the measurement of an operator” cannot in general depend only on the choice of that operator and on the system being measured (unless that system is in an eigenstate of that operator, or unless the operator itself is nondegenerate). The ambiguous relationship between Hermitian operators and physical observables complicates the epistemological meaning of the contradiction that we have found. On the other hand, in the original (additive) version of the Kochen-Specker theorem, we only had to count the number of positive answers in yes-no tests, and the physical meaning was clearer. In the present article, I shall show how to convert a multiplicative Kochen-Specker contradiction into an ordinary, additive one.

It may seem that such a conversion is trivial: just take the logarithms of the operators, and any product becomes a sum [9]. To find the logarithm of an operator, we first transform it to a basis where that operator is diagonal, we take the logarithms of its eigenvalues, and then we transform back to the original basis. In particular, if the eigenvalues are 1 and -1 , they become 0 and πi , respectively. Dividing the result by πi , we obtain a projection operator. However, if we sum up the projection operators corresponding to any row in array (1), we have a bad surprise: their product was the unit matrix, but the sum of the logarithms does not vanish! In some cases it may be $2\pi i$. With some hindsight, this could have been expected: a logarithm is a multiply valued function, defined only modulo $2\pi i$.

There is however a way of overcoming this hurdle: instead of requiring subsets of commuting projection operators to sum up to the unit matrix, we merely require their sum to have all its eigenvalues with the *same parity* (all even, or all odd). Since a “measurement” of this sum, if performed, must yield one of the eigenvalues, this means, in terms of our set of yes-no questions, that the number of positive answers must have a definite parity. This may again lead to contradictions.

Let us illustrate this with the nine operators in the above array. Their eigenvalues are ± 1 , and each one of them can be converted into a projection operator by the transformation $\Omega \rightarrow (1 - \Omega)/2$. This is equivalent to taking the logarithm and dividing by πi . For example, the first column of array (1) gives the three projection operators:

$$P_{2z} = (1 - \sigma_{2z})/2, \quad (3)$$

$$P_{1x} = (1 - \sigma_{1x})/2, \quad (4)$$

$$P_{1x2z} = (1 - \sigma_{1x}\sigma_{2z})/2, \quad (5)$$

where 1 now means the 4-dimensional unit matrix, and we have discarded the symbols $\mathbf{1} \otimes$ and $\otimes \mathbf{1}$, for brevity. Similar abbreviated notations will also be used in the sequel.

We obtain

$$P_{2z} + P_{1x} + P_{1x2z} = 2 - (1 + \sigma_{2z})(1 + \sigma_{1x})/2. \quad (6)$$

Since all these operators commute, and each parenthesis on the right hand side of (6) has eigenvalues 0 and 2, the eigenvalues of the entire left hand side also are 0 and 2. This agrees with the fact that the product of the three operators in the first column of our array is equal to 1, so that its logarithm is 0 (mod $2\pi i$).

If we now attempt to attach to each one of these projection operators a hypothetical numerical value, $v(P_{...}) = 0$ or 1, we have

$$v(P_{2z}) + v(P_{1x}) + v(P_{1x2z}) = 0 \quad \text{or} \quad 2. \quad (7)$$

In the same way, we find, for the second column,

$$v(P_{1z}) + v(P_{2x}) + v(P_{1z2x}) = 0 \quad \text{or} \quad 2, \quad (8)$$

and for the first two rows,

$$v(P_{2z}) + v(P_{1z}) + v(P_{1z2z}) = 0 \quad \text{or} \quad 2, \quad (9)$$

and

$$v(P_{1x}) + v(P_{2x}) + v(P_{1x2x}) = 0 \quad \text{or} \quad 2. \quad (10)$$

For the third row, we have

$$P_{1x2z} + P_{1z2x} + P_{1y2y} = 2 - (1 + \sigma_{1x} \sigma_{2z} + \sigma_{1z} \sigma_{2x} + \sigma_{1y} \sigma_{2y})/2, \quad (11)$$

$$= 2 - (1 + \sigma_{1x} \sigma_{2z})(1 + \sigma_{1z} \sigma_{2x})/2. \quad (12)$$

Again, the two parentheses on the right hand side commute, and each one has eigenvalues 0 and 2. We thus have

$$v(P_{1x2z}) + v(P_{1z2x}) + v(P_{1y2y}) = 0 \quad \text{or} \quad 2. \quad (13)$$

Finally, for the third column of array (1), we have

$$P_{1x2x} + P_{1y2y} + P_{1z2z} = 2 - (1 + \sigma_{1x} \sigma_{2x} + \sigma_{1y} \sigma_{2y} + \sigma_{1z} \sigma_{2z})/2. \quad (14)$$

The eigenvalues of the rotationally invariant operator $\sigma_{1x} \sigma_{2x} + \sigma_{1y} \sigma_{2y} + \sigma_{1z} \sigma_{2z}$ are well known: they are 1 for the triplet state, and -3 for the singlet state. We thus have

$$v(P_{1x2x}) + v(P_{1y2y}) + v(P_{1z2z}) = 1 \quad \text{or} \quad 3. \quad (15)$$

The contradiction is now obvious: on the left hand sides of Eqs. (7–10), (13), and (15), each one of the numbers $v(P_{...}) = 0$ or 1 appears twice. The sum of these left hand sides thus is an *even* number. On the other hand, the sum of the right hand sides is odd. We thus obtain a Kochen-Specker contradiction with nine projection operators in a 4-dimensional space.

The above contradiction is an algebraic property of these nine operators, irrespective of the quantum state of the physical system. However, if it is known that the latter has been prepared in a particular quantum state, for example the singlet state, a contradiction may be obtained with fewer operators [11]. Consider those in the first two columns of our array, so that Eqs. (7) and (8) still hold. For a singlet state, we also have

$$(\sigma_{1j} + \sigma_{2j}) \psi = 0, \quad (16)$$

and therefore

$$v(P_{1x}) + v(P_{2x}) = 1, \quad (17)$$

$$v(P_{1z}) + v(P_{2z}) = 1. \quad (18)$$

Moreover, for a singlet

$$(\sigma_{1x} \sigma_{2z} + \sigma_{1z} \sigma_{2x}) \psi = 0, \quad (19)$$

as may easily be verified by placing a factor

$$1 \equiv (\sigma_{1x} \sigma_{2z})^{-1} (\sigma_{1x} \sigma_{2z}), \quad (20)$$

on the left of Eq. (19). Therefore, the hypothetical values of the corresponding projection operators must satisfy

$$v(P_{1x2z}) + v(P_{1z2x}) = 1, \quad (21)$$

just as in Eqs. (17) and (18).

Consider now Eqs. (7), (8), (17), (18), and (21). On their left hand sides, each one of the numbers $v(P_{...}) = 0$ or 1 appears twice. The sum of these left hand sides is an *even* number, just as before, while the sum of their right hand sides is odd. This is a Kochen-Specker contradiction involving only six projection operators in a 4-dimensional space. It is however restricted to a particular singlet state.

I am grateful to N. D. Mermin for patiently explaining to me that ref. [11] was a Kochen-Specker argument, not one about locality, as I had wrongly thought. This work was supported by the Gerard Swope Fund, and the Fund for Encouragement of Research.

1. S. Kochen and E. P. Specker, J. Math. Mech. 17 (1967) 59.
2. M. Redhead, Incompleteness, nonlocality, and realism (Clarendon Press, Oxford, 1987).
3. R. Clifton, Am. J. Phys. 61 (1993) 443.
4. H. B. Johansen, Am. J. Phys. 62 (1994) 471.
5. A. Peres, J. Phys. A: Math. Gen. 24 (1991) L175.
6. A. Peres, Quantum theory: concepts and methods (Kluwer, Dordrecht, 1993) p. 114.
7. M. Kernaghan, J. Phys. A: Math. Gen. 27 (1994) L829.
8. M. Kernaghan and A. Peres, Phys. Letters A 198 (1995) 1.
9. A. Fine and P. Teller, Found. Phys. 8 (1978) 629.
10. N. D. Mermin, Phys. Rev. Lett. 65 (1990) 3373; Rev. Mod. Phys. 65 (1993) 803.
11. A. Peres, Phys. Letters A 151 (1990) 107; Found. Phys. 22 (1992) 357.